

Publ. Mat. **49** (2005), 285–296

LIPSCHITZ SPACES AND CALDERÓN-ZYGMUND OPERATORS ASSOCIATED TO NON-DOUBLING MEASURES

JOSÉ GARCÍA-CUERVA AND A. EDUARDO GATTO

Abstract

In the setting of a metric measure space (\mathbb{X}, d, μ) with an n -dimensional Radon measure μ , we give a necessary and sufficient condition for the boundedness of Calderón-Zygmund operators associated to the measure μ on Lipschitz spaces on the support of μ . Also, for the Euclidean space \mathbb{R}^d with an arbitrary Radon measure μ , we give several characterizations of Lipschitz spaces on the support of μ , $\mathcal{Lip}(\alpha, \mu)$, in terms of mean oscillations involving μ . This allows us to view the “regular” BMO space of X. Tolsa as a limit case for $\alpha \rightarrow 0$ of the spaces $\mathcal{Lip}(\alpha, \mu)$.

1. Introduction

The present paper is devoted, on the one hand, to study the invariance of Lipschitz spaces under Calderón-Zygmund operators associated to an n -dimensional Radon measure μ .

We do that in Section 2, in the context of a *metric measure space* (\mathbb{X}, d, μ) with an n -dimensional measure, that is a measure satisfying condition (2.1). This allows, in particular, non-doubling measures.

A second aim is to show that, for any Radon measure in the Euclidean space \mathbb{R}^d , the Lipschitz spaces can be characterized by a host of integral oscillation conditions similar to the regular BMO condition introduced by Tolsa. This shows that the regular BMO space of Tolsa is a limit case of the natural Lipschitz spaces associated to the measure.

The study of Calderón-Zygmund operators associated to an n -dimensional Radon measure was carried out, in the Lebesgue spaces, by

2000 *Mathematics Subject Classification.* 42B20, 26B35, 47B38, 47G10.

Key words. Calderón-Zygmund theory, singular integrals, Lipschitz spaces, BMO , non-doubling measures.

Supported in part by DGES, Spain, under grant BFM2001-0189. It is a pleasure for the second author, to thank the members of the Mathematics Department of Universidad Autónoma de Madrid for their friendly hospitality.

Nazarov, Treil and Volberg (see [NTV1], [NTV2]) and also by Tolsa (see [To1], [To2]). Further results, dealing with BMO and H^1 and providing boundedness criteria in the spirit of the $T(1)$ or $T(b)$ theorems, were obtained as well (see [NTV3], [MMNO], [To3]).

In [GG] we have also studied, on metric spaces, the theory of fractional integral operators associated to an n -dimensional Radon measure μ on Lebesgue spaces and Lipschitz spaces.

In a previous version of this paper we were only considering n -dimensional Radon measures, which are the ones we are mainly interested on and the only ones for which we can prove the boundedness of Calderón-Zygmund operators. However, Xavier Tolsa made the observation, that we gratefully acknowledge, that Theorem 3.3 and its proof were valid for a general Radon measure, since the n -dimensional nature of the measure, was never used.

We also want to thank Professor Peter Constantin and the referee for appropriate questions about Theorem 2.5 that helped us to obtain the present statement.

2. Calderón-Zygmund operators

In this section, (\mathbb{X}, d, μ) will be a *metric measure space* (that is, d is a distance on \mathbb{X} and μ is a Borel measure on \mathbb{X}), such that, for every ball

$$B(x, r) = \{y \in \mathbb{X} : d(x, y) < r\}, \quad x \in \mathbb{X}, r > 0,$$

we have

$$(2.1) \quad \mu(B(x, r)) \leq Cr^n,$$

where n is some fixed positive real number and C is independent of x and r .

We shall also refer to condition (2.1) by saying that the measure μ is n -dimensional.

Whenever we refer to “the ball B ”, we shall understand that we have chosen for it a fixed center and a fixed radius. That way, it makes sense to say that if B is a ball and k is a positive real number, we shall denote by kB the ball having the same center as B and radius k times that of B .

From now on, we shall assume that $\mu(\mathbb{X}) = \infty$.

We shall use below two basic lemmas from Section 2 of [GG], which allow us to bound the integrals against an n -dimensional measure of potential kernels on balls or complements of balls. For completeness, we group here in a single statement without proofs the two lemmas and a third property of the measure which will also be important in the sequel.

Lemma 2.1. *Let μ be an n -dimensional measure on (\mathbb{X}, d, μ) , $\gamma > 0$ and $r > 0$. Then*

- a)
$$\int_{B(x,r)} \frac{1}{d(x,y)^{n-\gamma}} d\mu(y) \leq Cr^\gamma$$
- b)
$$\int_{\mathbb{X} \setminus B(x,r)} \frac{1}{d(x,y)^{n+\gamma}} d\mu(y) \leq Cr^{-\gamma} \quad \text{and}$$
- c)
$$\int_{r/2 \leq d(x,y) < r} \frac{1}{d(x,y)^n} d\mu(y) \leq C,$$

where, in all cases, C is a constant independent of r .

Definition 2.2. Given α , $0 < \alpha < 1$, we shall say that a function defined on the support of μ , $f: \text{supp}(\mu) \rightarrow \mathbb{C}$ is a Lipschitz function of order α when

$$(2.2) \quad |f(x) - f(y)| \leq Cd(x,y)^\alpha \text{ for every } x, y \in \text{supp}(\mu)$$

and the smallest constant in inequality (2.2) will be denoted $\|f\|_{\mathcal{Lip}(\alpha, \mu)}$. The linear space of Lebesgue classes of Lipschitz functions of order α , modulo constants, becomes a Banach space with the norm $\|\cdot\|_{\mathcal{Lip}(\alpha, \mu)}$; it will be denoted $\mathcal{Lip}(\alpha, \mu)$.

Note also that if a function defined μ -almost everywhere on \mathbb{X} satisfies (2.2) μ -a.e., then it coincides μ -a.e. with a Lipschitz function on the support of μ .

Next, we define the class of singular kernels that we consider in this paper.

Definition 2.3. A singular kernel on a metric measure space (\mathbb{X}, d, μ) with μ n -dimensional, will be a measurable function $K(x, y)$ on $\mathbb{X} \times \mathbb{X} \setminus \{x = y\}$ satisfying the following conditions:

- (1) $|K(x, y)| \leq \frac{A_1}{d(x, y)^n}$.
- (2) $|K(x_1, y) - K(x_2, y)| \leq \frac{A_2 d(x_1, x_2)^\delta}{d(x_1, y)^{n+\delta}}$ for $2d(x_1, x_2) \leq d(x_1, y)$, where δ , $0 < \delta \leq 1$ is a regularity constant specific to the kernel.
- (3) $\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < d(x, y) < 1} K(x, y) d\mu(y)$ exists for μ -almost every point x .

With K we associate the truncated kernels

$$K_\varepsilon(x, y) = K(x, y)\chi_{\{d(x, y) > \varepsilon\}}(x, y).$$

Finally, through the truncated kernels, we are in a position to introduce the singular integral operators that will be our object of study in this article.

Definition 2.4. For $f \in \mathcal{Lip}(\alpha, \mu)$, $0 < \alpha < \delta \leq 1$, we define

$$\tilde{T}_\varepsilon f(x) = \int_{\mathbb{X}} (K_\varepsilon(x, y) - K_1(x_0, y)) f(y) \, d\mu(y),$$

where x_0 is a fixed point in \mathbb{X} and then we also define

$$\tilde{T}f(x) = \lim_{\varepsilon \rightarrow 0} \tilde{T}_\varepsilon f(x).$$

Note that it follows from the properties of $K(x, y)$ and Lemma 2.1 that the limit exists μ -almost everywhere. Indeed, for $\varepsilon < 1$

$$\begin{aligned} \tilde{T}_\varepsilon f(x) &= \int_{d(x, y) < 1} K_\varepsilon(x, y) (f(y) - f(x)) \, d\mu(y) \\ &\quad + \left(\int_{\varepsilon < d(x, y) < 1} K(x, y) \, d\mu(y) \right) f(x) \\ &\quad + \int_{\mathbb{X}} (K_1(x, y) - K_1(x_0, y)) f(y) \, d\mu(y), \end{aligned}$$

where the first and third integrals are absolutely convergent and the second term converges by property (3) of Definition 2.3.

Theorem 2.5. *Let K be a singular kernel as above and let \tilde{T} be the corresponding singular integral operator. Let $0 < \alpha < \delta \leq 1$. Then \tilde{T} is a bounded operator on $\mathcal{Lip}(\alpha, \mu)$ if and only if there are constants B_1 and B_2 such that*

$$(a) \quad \tilde{T}(1)(x) = B_1 \quad \mu\text{-a.e.}$$

and

$$(b) \quad \left| \int_{r < d(x, y) < R} K(x, y) \, d\mu(y) \right| \leq B_2, \text{ for all } 0 < r < R \text{ and } \mu\text{-a.e. } x.$$

Proof: We shall show first that conditions (a) and (b) are sufficient.

Except for a set of μ -measure zero that depends on $K(x, y)$ and μ , we have

$$\begin{aligned} \tilde{T}f(x_1) - \tilde{T}f(x_2) &= \lim_{\varepsilon \rightarrow 0} \left\{ \tilde{T}_\varepsilon f(x_1) - \tilde{T}_\varepsilon f(x_2) \right\} \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{X}} (K_\varepsilon(x_1, y) - K_\varepsilon(x_2, y)) f(y) \, d\mu(y). \end{aligned}$$

The same computation with $f = 1$ and condition (a) imply that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{X}} (K_\varepsilon(x_1, y) - K_\varepsilon(x_2, y)) \, d\mu(y) = 0.$$

Therefore

$$\tilde{T}f(x_1) - \tilde{T}f(x_2) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{X}} (K_\varepsilon(x_1, y) - K_\varepsilon(x_2, y)) (f(y) - f(x_1)) \, d\mu(y).$$

Now let $r = d(x_1, x_2)$ and take $\varepsilon < r$. After splitting the integral in the limit above as the sum of the integral over $B(x_1, 3r)$ and the integral over $\mathbb{X} \setminus B(x_1, 3r)$, we can write

$$\begin{aligned} & \tilde{T}f(x_1) - \tilde{T}f(x_2) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{B(x_1, 3r)} K_\varepsilon(x_1, y) (f(y) - f(x_1)) \, d\mu(y) \\ & \quad - \lim_{\varepsilon \rightarrow 0} \int_{B(x_1, 3r)} K_\varepsilon(x_2, y) (f(y) - f(x_2)) \, d\mu(y) \\ & \quad - (f(x_2) - f(x_1)) \lim_{\varepsilon \rightarrow 0} \int_{B(x_2, 2r)} K_\varepsilon(x_2, y) \, d\mu(y) \\ & \quad - (f(x_2) - f(x_1)) \lim_{\varepsilon \rightarrow 0} \int_{B(x_1, 3r) \setminus B(x_2, 2r)} K_\varepsilon(x_2, y) \, d\mu(y) \\ & \quad + \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{X} \setminus B(x_1, 3r)} (K_\varepsilon(x_1, y) - K_\varepsilon(x_2, y)) (f(y) - f(x_1)) \, d\mu(y) \\ &= I_1 - I_2 - I_3 - I_4 + I_5. \end{aligned}$$

Observe now that, by combining condition (1) of Definition 2.3 together with the Lipschitz condition for f and using part a) of Lemma 2.1, the integral in I_1 converges absolutely and $|I_1| \leq C_1 \|f\|_{\mathcal{L}ip(\alpha, \mu)} r^\alpha$. Then, after realizing that $B(x_1, 3r) \subset B(x_2, 4r)$, we can use the same argument to see that the integral in I_2 also converges absolutely and, we have $|I_2| \leq C_2 \|f\|_{\mathcal{L}ip(\alpha, \mu)} r^\alpha$. To control I_3 we use condition (b) to obtain $|I_3| \leq B_2 \|f\|_{\mathcal{L}ip(\alpha, \mu)} r^\alpha$. Next, we can use part c) of Lemma 2.1 to prove that the integral in I_4 converges absolutely and we have $|I_4| \leq C_4 \|f\|_{\mathcal{L}ip(\alpha, \mu)} r^\alpha$.

Finally, using condition (2) in Definition 2.3, and part b) of Lemma 2.1, we see that the integral in I_5 converges absolutely and $|I_5| \leq C_5 \|f\|_{\mathcal{L}ip(\alpha, \mu)} r^\alpha$. This completes the proof of the sufficiency.

Next we shall show that the conditions are necessary. First of all, we observe that condition (a) follows from the fact that $\|1\|_{\mathcal{L}ip(\alpha, \mu)} = 0$.

To prove (b) consider first $r = d(x, x_2)$, $x, x_2 \in \text{supp}(\mu)$ such that the limit in condition (3) of Definition 2.3 exists both for x and x_2 . Let

$f(x) = d(x, x_2)^\alpha$. From the decomposition we made above in the proof of the sufficiency, we can write

$$\begin{aligned} \left(\tilde{T}(f)(x) - \tilde{T}(f)(x_2) \right) - I_1 + I_2 + I_4 - I_5 &= -I_3 \\ &= d(x, x_2)^\alpha \lim_{\varepsilon \rightarrow 0} \int_{d(x_2, y) < 2r} K_\varepsilon \, d\mu(y). \end{aligned}$$

Since the left hand side is less than or equal to $B d(x, x_2)^\alpha$ with a constant B independent of x , we obtain

$$\left| \lim_{\varepsilon \rightarrow 0} \int_{d(x_2, y) < 2r} K_\varepsilon(x_2, y) \, d\mu(y) \right| \leq B$$

with $r = d(x, x_2)$, μ -a.e. in $\text{supp}(\mu)$. Since μ is n -dimensional, we also have

$$(2.3) \quad \left| \lim_{\varepsilon \rightarrow 0} \int_{d(x_2, y) < r} K_\varepsilon(x_2, y) \, d\mu(y) \right| \leq B'$$

with $r = d(x, x_2)$, μ -a.e. in $\text{supp}(\mu)$.

Now, it is easy to see that condition (2.3) holds for all $r > 0$. Indeed, let $r > 0$ and assume that $d(x_2, x) \neq r$, for $x \in \text{supp}(\mu)$ except for a fixed set E of measure 0. If there is no $\bar{x} \in \text{supp}(\mu) \setminus E$ with $r/2 \leq d(x_2, \bar{x}) < r$, then $\int_{r/2 \leq d(x_2, y) < r} K_\varepsilon(x_2, y) \, d\mu(y) = 0$; if there is \bar{x} satisfying $d(x_2, \bar{x}) = s$ with $r/2 \leq s < r$, we have

$$\begin{aligned} \left| \lim_{\varepsilon \rightarrow 0} \int_{d(x_2, y) < r} K_\varepsilon(x_2, y) \, d\mu(y) \right| &\leq \left| \lim_{\varepsilon \rightarrow 0} \int_{d(x_2, y) < s} K_\varepsilon(x_2, y) \, d\mu(y) \right| \\ &\quad + \int_{r/2 \leq d(x_2, y) < r} |K_\varepsilon(x_2, y)| \, d\mu(y) \\ &\leq B + C. \end{aligned}$$

Finally, we can get condition (b). Let $0 < r < R$. We have

$$\begin{aligned} \int_{r < d(x_2, y) < R} K_\varepsilon(x_2, y) \, d\mu(y) &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < d(x_2, y) < R} K(x_2, y) \, d\mu(y) \\ &\quad - \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < d(x_2, y) \leq r} K(x_2, y) \, d\mu(y), \end{aligned}$$

which implies (b) with constant B_2 independent of x_2 , r and R . \square

Remark 2.6. Conditions (1) and (2) in Definition 2.3 are traditionally called “standard estimates” in the literature on singular integrals. Condition (3) is also classical for singular integrals of principal value type. Therefore, we can say that Theorem 2.5 is valid for all standard singular integrals of principal value type.

Condition (b) in Theorem 2.5 is a weak cancellation condition that is also present in the classical literature on singular integrals on Euclidean spaces with Lebesgue measure dating back to Calderón and Zygmund. It can be seen, for instance, in [BCP]. It has also appeared recently in the context of general measures in the work of P. Mattila. See, for example [M2], where it is used, for the Riesz kernels, to derive some local rectifiability properties of the measure.

3. Characterization of Lipschitz spaces

All throughout this section, μ will be a fixed Radon measure on \mathbb{R}^d . This is all that we need for the results we prove in this section, in particular for Theorem 3.3. Only for Remark 3.8 we will need to assume that the measure is n -dimensional.

From now on, all balls that we consider will be centered at points in the support of μ .

In order to prove the main theorem of this section we will need the following known definition and lemma (see [To3]).

Definition 3.1. Let β be a fixed constant. A ball B is called β -doubling if

$$\mu(2B) \leq \beta\mu(B).$$

Lemma 3.2. Let $f \in L^1_{\text{loc}}(\mu)$. If $\beta > 2^d$, then, for almost every x with respect to μ , there exists a sequence of β -doubling balls $B_j = B(x, r_j)$ with $r_j \rightarrow 0$, such that

$$\lim_{j \rightarrow \infty} \frac{1}{\mu(B_j)} \int_{B_j} f(y) \, d\mu(y) = f(x).$$

Proof: We will show that for almost every x with respect to μ there is a β -doubling ball centered at x with radius as small as we wish. This fact, combined with the differentiation theorem, completes the proof of the lemma.

We know that for almost every x with respect to μ

$$(3.1) \quad \lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^d} > 0$$

(differentiation of μ with respect to Lebesgue measure, see [M1]). Now for x satisfying (3.1) take $B = B(x, r)$ and assume that none of the balls $2^{-k}B$, $k \geq 1$, is β -doubling. Then it is easy to see that $\mu(B) > \beta^k \mu(2^{-k}B)$ for all $k \geq 1$. Therefore

$$\frac{\mu(2^{-k}B)}{(2^{-k}r)^d} < \left(\frac{2^d}{\beta}\right)^k \frac{\mu(B)}{r^d}.$$

Note that, since $\beta > 2^d$, the right hand side tends to zero for $k \rightarrow \infty$, which is a contradiction. \square

Now we can state and prove the main result of this section.

Theorem 3.3. *For a function $f \in L^1_{\text{loc}}(\mu)$, the conditions I, II, and III below, are equivalent*

- (I) *There exist some constant C_1 and a collection of numbers f_B , one for each ball B , such that these two properties hold: For any ball B with radius r*

$$(3.2) \quad \frac{1}{\mu(2B)} \int_B |f(x) - f_B| \, d\mu(x) \leq C_1 r^\alpha,$$

and for any ball U such that $B \subset U$ and $\text{radius}(U) \leq 2r$,

$$(3.3) \quad |f_B - f_U| \leq C_1 r^\alpha.$$

- (II) *There is a constant C_2 such that*

$$(3.4) \quad |f(x) - f(y)| \leq C_2 |x - y|^\alpha$$

for μ -almost every x and y in the support of μ .

- (III) *For any given p , $1 \leq p \leq \infty$, there is a constant $C(p)$, such that for every ball B of radius r , we have*

$$(3.5) \quad \left(\frac{1}{\mu(B)} \int_B |f(x) - m_B(f)|^p \, d\mu(x) \right)^{1/p} \leq C(p) r^\alpha,$$

where $m_B(f) = \frac{1}{\mu(B)} \int_B f(y) \, d\mu(y)$ and also for any ball U such that $B \subset U$ and $\text{radius}(U) \leq 2r$,

$$(3.6) \quad |m_B(f) - m_U(f)| \leq C(p) r^\alpha.$$

In addition, the quantities: $\inf C_1$, $\inf C_2$, and $\inf C(p)$ with a fixed p are equivalent.

Proof: (I) \Rightarrow (II). Consider x as in the lemma and let $B_j = B(x, r_j)$, $j \geq 1$, a sequence of β -doubling balls with $r_j \rightarrow 0$. We will show first that (3.2) implies

$$\lim_{j \rightarrow \infty} f_{B_j} = f(x).$$

It suffices to observe that

$$\begin{aligned} |m_{B_j}(f) - f_{B_j}| &\leq \frac{1}{\mu(B_j)} \int_{B_j} |f(y) - f_{B_j}| \, d\mu(y) \\ &\leq \frac{\mu(2B_j)}{\mu(B_j)} \frac{1}{\mu(2B_j)} \int_{B_j} |f(y) - f_{B_j}| \, d\mu(y) \leq \beta C_1 r_j^\alpha. \end{aligned}$$

Next, let x and y be two points as in the lemma. Take $B = B(x, r)$ any ball with $r \leq |x - y|$ and let $U = B(x, 2|x - y|)$. Now define $B_k = B(x, 2^k r)$, for $0 \leq k \leq \bar{k}$, where \bar{k} is the first integer such that $2^{\bar{k}} r \geq |x - y|$. Then

$$\begin{aligned} |f_B - f_U| &\leq \sum_{k=0}^{\bar{k}-1} |f_{B_k} - f_{B_{k+1}}| + |f_{B_{\bar{k}}} - f_U| \\ &\leq C_1 \sum_{k=0}^{\bar{k}} (2^k r)^\alpha \leq C' C_1 |x - y|^\alpha, \end{aligned}$$

where C' is independent of x and B .

A similar argument can be made for the point y with any ball $B' = B(y, s)$ such that $s \leq |x - y|$ and $V = B(y, 3|x - y|)$. Therefore

$$|f_B - f_{B'}| \leq |f_B - f_U| + |f_U - f_V| + |f_V - f_{B'}| \leq C'' C_1 |x - y|^\alpha.$$

Finally, take two sequences of β -doubling balls $B_j = B(x, r_j)$ and $B'_j = B(y, s_j)$ with $r_j \rightarrow 0$ and $s_j \rightarrow 0$. We have

$$|f(x) - f(y)| = \lim_{j \rightarrow \infty} |f_{B_j} - f_{B'_j}| \leq C'' C_1 |x - y|^\alpha.$$

(II) \Rightarrow (III). It is immediate. Note also that (II) \Rightarrow (I) is immediate as well.

(III) \Rightarrow (I). Define first $f_B = m_B(f)$. Then (3.3) is exactly (3.6).

In addition, the left hand side of (3.2) is less than or equal to the left hand side of (3.5).

This concludes the proof of the theorem. \square

Remark 3.4. Theorem 3.3 is also true if the number 2 in condition (I) is replaced by any fixed $\rho > 1$. In that case, the proof uses (ρ, β) -doubling balls, that is, balls satisfying $\mu(\rho B) \leq \beta \mu(B)$. However this extension is not needed in our paper.

The idea of combining the mean oscillation condition with an extra condition as in (I) originates in the work of Tolsa [To3] on regular BMO , whereas the introduction of the ρ factor in (I) comes from [NTV3].

Definition 3.5. We shall call Lipschitz function of order α with respect to μ to a function, or rather the corresponding Lebesgue class in $L^1_{\text{loc}}(\mu)$, which satisfies any, and hence all, of the conditions of Theorem 3.3.

The linear space of all Lipschitz functions of order α , with respect to μ , modulo constants, becomes, with the norm $\inf C_2$ of Theorem 3.3, a Banach space, which we shall call $\mathcal{Lip}(\alpha, \mu)$.

Remark 3.6. It is easy to see that $\mathcal{Lip}(\alpha, \mu)$ coincides with the space of Lipschitz functions of order α on the support of μ .

Note that by the extension theorem of Banach (see [B] or [Mi]¹), any Lipschitz function of order α with respect to μ has an extension to \mathbb{R}^d that is a Lipschitz function of order α with an equivalent norm.

Remark 3.7. For $0 < \alpha \leq 1$, a telescoping argument like the one used in the proof of (I) \Rightarrow (II) in Theorem 3.3 shows that (3.3) is equivalent to

$$(3.7) \quad |f_B - f_U| \leq C'_1 \text{radius}(U)^\alpha$$

for any two balls $B \subset U$.

Remark 3.8. For this remark we further assume that μ is n -dimensional. Then (3.7) is also equivalent to

$$(3.8) \quad |f_B - f_U| \leq C''_1 K_{B,U} \text{radius}(U)^\alpha,$$

for any two balls $B \subset U$, where $K_{B,U}$ is the constant introduced by X. Tolsa in [To3], given by

$$K_{B,U} = 1 + \sum_{j=1}^{N_{B,U}} \frac{\mu(2^j B)}{(2^j r)^n},$$

with $N_{B,U}$ equal to the first integer k such that $2^k \text{radius}(B) \geq \text{radius}(U)$. Indeed (3.8) for comparable balls, that is, for $\text{radius}(U) \leq 2 \text{radius}(B)$, reduces to (3.3) because, in this case, $K_{B,U}$ is controlled by an absolute constant.

Note that (3.2) and (3.8) make sense also for $\alpha = 0$ and the space defined by them is the space $RBMO(\mu)$ of X. Tolsa (see [To3]). Therefore, the spaces $\mathcal{Lip}(\alpha, \mu)$, $0 < \alpha \leq 1$ can be seen as members of a family containing also $RBMO(\mu)$.

¹In this reference, there is a note added in proof, to the effect that “Banach’s theorem mentioned above is probably due to McShane”.

References

- [B] S. BANACH, “*Wstęp do teorii funkcji rzeczywistych*”, [Introduction to the theory of real functions], Monografie Matematyczne **XVII**, Polskie Towarzystwo Matematyczne, Warszawa-Wrocław, 1951.
- [BCP] A. BENEDEK, A.-P. CALDERÓN AND R. PANZONE, Convolution operators on Banach space valued functions, *Proc. Nat. Acad. Sci. U.S.A.* **48** (1962), 356–365.
- [D] G. DAVID, “*Wavelets and singular integrals on curves and surfaces*”, Lecture Notes in Mathematics **1465**, Springer-Verlag, Berlin, 1991.
- [GG] J. GARCÍA-CUERVA AND A. E. GATTO, Boundedness properties of fractional integral operators associated to non-doubling measures, *Studia Math.* **162(3)** (2004), 245–261.
- [MS1] R. A. MACÍAS AND C. SEGOVIA, Singular integrals on generalized Lipschitz and Hardy spaces, *Studia Math.* **65(1)** (1979), 55–75.
- [MS2] R. A. MACÍAS AND C. SEGOVIA, Lipschitz functions on spaces of homogeneous type, *Adv. in Math.* **33(3)** (1979), 257–270.
- [MMNO] J. MATEU, P. MATTILA, A. NICOLAU AND J. OROBITG, BMO for nondoubling measures, *Duke Math. J.* **102(3)** (2000), 533–565.
- [M1] P. MATTILA, “*Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*”. Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press, Cambridge, 1995.
- [M2] P. MATTILA, Singular integrals and rectifiability, in: “*Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations*” (El Escorial, 2000), *Publ. Mat.* **Vol. Extra** (2002), 199–208.
- [Me] Y. MEYER, “*Ondelettes et opérateurs. II. Opérateurs de Calderón-Zygmund*”, [Wavelets and operators. II. Calderón-Zygmund operators], Actualités Mathématiques, [Current Mathematical Topics], Hermann, Paris, 1990.
- [Mi] G. J. MINTY, On the extension of Lipschitz, Lipschitz-Hölder continuous, and monotone functions, *Bull. Amer. Math. Soc.* **76** (1970), 334–339.

- [NTV1] F. NAZAROV, S. TREIL AND A. VOLBERG, Cauchy integral and Calderón-Zygmund operators on nonhomogeneous spaces, *Internat. Math. Res. Notices* **15** (1997), 703–726.
- [NTV2] F. NAZAROV, S. TREIL AND A. VOLBERG, Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces, *Internat. Math. Res. Notices* **9** (1998), 463–487.
- [NTV3] F. NAZAROV, S. TREIL AND A. VOLBERG, Accretive system Tb -theorems on nonhomogeneous spaces, *Duke Math. J.* **113**(2) (2002), 259–312.
- [S] E. M. STEIN, “*Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*”, Princeton Mathematical Series **43**, Monographs in Harmonic Analysis III, Princeton University Press, Princeton, NJ, 1993.
- [To1] X. TOLSA, L^2 -boundedness of the Cauchy integral operator for continuous measures, *Duke Math. J.* **98**(2) (1999), 269–304.
- [To2] X. TOLSA, Cotlar’s inequality without the doubling condition and existence of principal values for the Cauchy integral of measures, *J. Reine Angew. Math.* **502** (1998), 199–235.
- [To3] X. TOLSA, BMO, H^1 , and Calderón-Zygmund operators for non doubling measures, *Math. Ann.* **319**(1) (2001), 89–149.

José García-Cuerva:
 Departamento de Matemáticas, C-XV
 Universidad Autónoma
 28049, Madrid
 Spain
E-mail address: jose.garcia-cuerva@uam.es

A. Eduardo Gatto:
 Department of Mathematics
 DePaul University
 Chicago, Illinois, 60614
 U. S. A.
E-mail address: aegatto@condor.depaul.edu

Primera versió rebuda el 17 de març de 2004,
 darrera versió rebuda el 2 de febrer de 2005.